Cubic Trigonometric Hermite Parametric Curves and Surfaces with Shape Parameters

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Abstract

A new kind of cubic trigonometric Hermite parametric curves and surfaces analogous to the normal cubic Hermite parametric curves and surfaces, with shape parameters, are presented in this work. The new curves and surfaces not only inherit properties of the normal cubic Hermite curves and surfaces in polynomial space respectively, but also enjoy some other advantageous properties for modeling. For given boundary conditions, the shapes of the new curves and surfaces can be adjusted by using the shape parameters with them. When the boundary conditions are chosen appropriately, the presented curves can exactly represent some engineering curves.

Keywords

Trigonometric Polynomial; Hermite Curve; Hermite Surface; Shape Parameter

Introduction

In Computer Aided Geometric Design and CAD/CAM, curves and surfaces are generally constructed by polynomial functions such as Ferguson curve, Coons patch, Bézier curve and surface, B-spline curve and surface. To extend the expression forms of polynomial curves and surfaces, trigonometric polynomials have received very munch attention in recent years. For instance, Zhang constructed C-Ferguson curve, C-Bézier curve and C-B-spline curve in the space {1, t, sin$t$, cos$t$} [1]. Mainar and Chen defined C- Bézier curves of higher order in the space {1, t, …, t^{k-3}, cos$t$, sin$t$} [2,3]. In the same space, Wang constructed non-uniform algebraic trigonometric B-splines [4]. Han presented a cubic trigonometric Bézier curve with two shape parameters in the space {1, sin$t$, cos$t$, sin^2t} [5]. In the same space, a family of quasi-cubic trigonometric curves was defined by Li [6]. Yan discussed a class of algebraic-trigonometric blended splines in the space {1, t, sin$t$, cos$t$, sin^2t, sin^3t, cos^3t} [7]. Liu constructed the biquadratic TC-Bézier curve in the space {1, sin$t$, cos$t$, sin2t, cos2t} [8]. Xie presented a class of Bézie-type curves based on the blending of algebraic and trigonometric polynomials [9]. Xie constructed a class of mixed Coons patch with shape parameters in the space {1, u, u^2, sin$u$, cos$u$} [10]. Zhong presented a class of algebraic-trigonometric cubic Hermite interpolating curve with a shape parameter [11]. These curves and surfaces constructed by trigonometric polynomials inherit most properties of the corresponding polynomial curves and surfaces, and some of them have other excellent abilities such as the character of shape adjustment and the exactly representation of some engineering curves and surfaces.

As useful interpolation models, cubic Hermite parametric curves and surfaces have been widely applied in CAGD. But there still exist several limitations of the normal cubic Hermite curves and surfaces, which limit their applications. First, when the boundary conditions are specified, the shapes of them cannot be adjusted. Second, the normal cubic Hermite curves cannot exactly represent some engineering curves such as ellipse and parabola. The purpose of this work is to present a class of cubic trigonometric Hermite parametric curves and surfaces, analogous to the corresponding normal cubic Hernite parametric curves and surfaces, with shape parameters.

The rest of this paper is organized as follows. In Section 2, the cubic trigonometric Hermite (CT-Hermite in short) basis functions are defined. In Section 3, the corresponding cubic trigonometric Hermite parametric curves (CT-Hermite curves in short) are established, and the representation of some engineering curves by CT-Hermite curves is discussed. In Section 4, the corresponding cubic trigonometric Hermite parametric surfaces (CT-Hermite surfaces in short) are defined. A short conclusion is given in Section 5.

Cubic Trigonometric Hermite Basis Functions

Definition 1. For $0 \le t \le \pi/2$, $\lambda_i, \mu_i \in R$, the following four functions

$$\begin{cases} F_i(t) = \lambda_i \sin^2 t - \lambda_i \sin^3 t + \cos^3 t \\ F_{i+1}(t) = 1 - \lambda_i \sin^2 t + \lambda_i \sin^3 t - \cos^3 t \\ G_i(t) = -\mu_i + \sin t + \mu_i \sin^2 t - \sin^3 t + \mu_i \cos^3 t \\ G_{i+1}(t) = -\cos t + \mu_i \sin^2 t - \mu_i \sin^3 t + \cos^3 t \end{cases} \quad (1)$$

are called cubic trigonometric Hermite (CT-Hermite in short) basis functions, where λ_i and μ_i are called shape parameters.

Simple calculations verify that, for arbitrary $\lambda_i, \mu_i \in R$, CT-Hermite basis functions satisfy

$$F_i(0) = 1,\ F_{i+1}(0) = 0,\ G_i(0) = 0,\ G_{i+1}(0) = 0,$$

$$F_i(\pi/2) = 0,\ F_{i+1}(\pi/2) = 1,\ G_i(\pi/2) = 0,\ G_{i+1}(\pi/2) = 0,$$

$$F_i'(0) = 0,\ F_{i+1}'(0) = 0,\ G_i'(0) = 1,\ G_{i+1}'(0) = 0,$$

$$F_i'(\pi/2) = 0,\ F_{i+1}'(\pi/2) = 0,\ G_i'(\pi/2) = 0,\ G_{i+1}'(\pi/2) = 1,$$

and $F_i(t) + F_{i+1}(t) = 1$, $G_i(t) = -G_{i+1}(\pi/2 - t)$.

The above results show that CT-Hermite basis functions have the same properties with normal cubic polynomial Hermite basis functions. However, CT-Hermite basis functions have the shape parameters λ_i and μ_i, different CT-Hermite basis functions can be got when λ_i and μ_i are of different values.

Cubic Trigonometric Hermite Parametric Curves

Definition and Properties of the Curves

Definition 2. Given a sequence of endpoints $\boldsymbol{p}_i$ and tangent vectors $\boldsymbol{p}_i'$ $(i = 0,1,2,\cdots,n-1)$, for $0 \le t \le \pi/2$, the following curves

$$\boldsymbol{H}_i(t) = F_i(t)\boldsymbol{p}_i + F_{i+1}(t)\boldsymbol{p}_{i+1} + G_i(t)\boldsymbol{p}_i' + G_{i+1}(t)\boldsymbol{p}_{i+1}' \quad (2)$$

$$(i = 0,1,2,\cdots,n-1)$$

are called segmented cubic trigonometric Hermite parametric curves (CT-Hermite curves in short), where $F_{i+j}(t)$ and $G_{i+j}(t)$ $(j = 0,1)$ are CT-Hermite basis functions defined as Eq. 1.

From the properties of CT-Hermite basis functions, we can simply calculate that the CT-Hermite curves satisfy

$$\begin{matrix} \boldsymbol{H}_i(\pi/2) = \boldsymbol{p}_{i+1} = \boldsymbol{H}_{i+1}(0) \\ \boldsymbol{H}_i'(\pi/2) = \boldsymbol{p}_{i+1}' = \boldsymbol{H}_{i+1}'(0) \end{matrix} \quad (i = 0,1,2,\cdots,n-2) \quad (3)$$

Eq. 3 shows that CT-Hermite curves interpolate in the endpoint and tangent vectors and satisfy C^1 continuous, which is the same with the normal cubic Hermite parametric curves [8]. Given the endpoints $\boldsymbol{p}_i$ and tangent vectors $\boldsymbol{p}_i'$ $(i = 0,1,2,\cdots,n-1)$, the shape of normal cubic Hermite curves cannot be changed, while the shape of CT-Hermite curves can be local or global adjusted by using the parameters λ_i and μ_i $(i = 0,1,2,\cdots,n-1)$ in the premise of ensuring C^1 continuous.

Example 1 Given a sequence of endpoints and tangent vectors are $\boldsymbol{p}_0 = (0,0)$, $\boldsymbol{p}_1 = (1,0)$, $\boldsymbol{p}_2 = (2,0)$, $\boldsymbol{p}_3 = (3,0)$, $\boldsymbol{p}_4 = (4,0)$, $\boldsymbol{p}_0' = (0,1)$, $\boldsymbol{p}_1' = (0,-1)$, $\boldsymbol{p}_2' = (0,1)$, $\boldsymbol{p}_3' = (0,-1)$, $\boldsymbol{p}_4' = (0,1)$, a entail curve satisfying C^1 continuous can be generated by four segments of CT-Hermite curves, and shape of the curve can be local or global adjusted by using the shape parameters λ_i and μ_i $(i = 0,1,2,3)$. Figure 1 shows local adjustment of the curve by λ_1, where $\lambda_i = \mu_j = 0$ $(i = 0,2,3; j = 0,1,2,3)$, dashed lines are $\lambda_1 = -1$, solid lines are $\lambda_1 = 0$, and dotted lines are $\lambda_1 = 1$. Figure 2 shows local adjustment of the curve by μ_2, where $\lambda_i = \mu_j = 0$ $(i = 0,1,2,3; j = 0,1,3)$, dashed lines are $\mu_2 = -1$, solid lines are $\mu_2 = 0$, and dotted lines are $\mu_2 = 1$. Let $\lambda_i = \lambda$, $\mu_i = \mu$ $(i = 0,1,2,3)$, Figure 3 shows global adjustment of the curve by λ, where $\mu = 0$, dashed lines are $\lambda = -1$, solid lines are $\lambda = 0$, and dotted lines are $\lambda = 1$. Figure 4 shows global adjustment of the curve by μ, where $\lambda = 0$, dashed lines are $\mu = -1$, solid lines are $\mu = 0$, and dotted lines are $\mu = 1$.

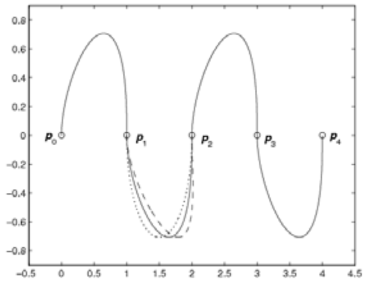

FIGURE 1 LOCAL ADJUSTMENT OF THE CURVE BY SHAPE PARAMETER λ_1

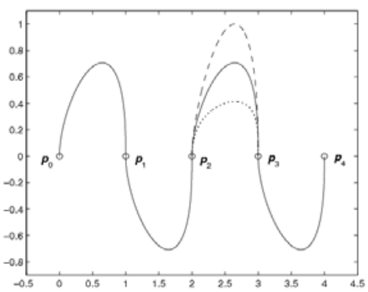

FIGURE 2 LOCAL ADJUSTMENT OF THE CURVE BY SHAPE PARAMETER μ_2

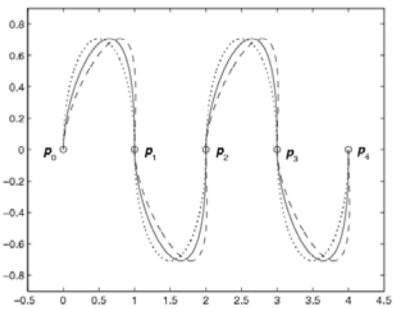

FIGURE 3 GLOBAL ADJUSTMENT OF THE CURVE BY SHAPE PARAMETER λ

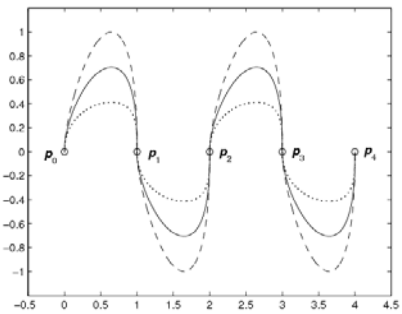

FIGURE 4 GLOBAL ADJUSTMENT OF THE CURVE BY SHAPE PARAMETER μ

Representation of Some Engineering Curves

Only one segment of CT-Hermite curves is discussed here. Given proper endpoints $\boldsymbol{p}_i$, $\boldsymbol{p}_{i+1}$ and tangent vectors $\boldsymbol{p}_i'$, $\boldsymbol{p}_{i+1}'$, the corresponding CT-Hermite curve segment $\boldsymbol{H}_i(t)$ can be used to represent some engineering curves exactly, such as line segment, ellipse, parabola, which is very difficult to achieve for the normal cubic Hermite curves.

1) Representation of a Line Segment

Theorem 1. Let the endpoints and tangent vectors are $\boldsymbol{p}_i = (a,0)$, $\boldsymbol{p}_{i+1} = (0,b)$, $\boldsymbol{p}_i' = \boldsymbol{p}_{i+1}' = (0,0)$, $ab \neq 0$, and shape parameters are $\lambda_i = \mu_i = 0$, the corresponding CT-Hermite curve segment $\boldsymbol{H}_i(t)$ represents a line segment.

Proof. Taking $\boldsymbol{p}_i = (a,0)$, $\boldsymbol{p}_{i+1} = (0,b)$, $\boldsymbol{p}_i' = \boldsymbol{p}_{i+1}' = (0,0)$ $(ab \neq 0)$ into Eq. 2 and let $\lambda_i = \mu_i = 0$, then the coordinates of CT-Hermite curve segment $\boldsymbol{H}_i(t)$ is

$$\begin{cases} x = a\cos^3 t \\ y = b(1-\cos^3 t) \end{cases}$$

This gives the intrinsic equation $\frac{x}{a} + \frac{y}{b} = 1$, it is an equation of a line segment.

2) Representation of an Ellipse

Theorem 2. Let the endpoints and tangent vectors are $\boldsymbol{p}_i = (a,0)$, $\boldsymbol{p}_{i+1} = (0,b)$, $\boldsymbol{p}_i' = (0,b)$, $\boldsymbol{p}_{i+1}' = (-a,0)$, $ab \neq 0$, and shape parameters are $\lambda_i = \mu_i = 1$, the corresponding CT-Hermite curve segment $\boldsymbol{H}_i(t)$ represents an ellipse.

Proof. Taking $\boldsymbol{p}_i = (a,0)$, $\boldsymbol{p}_{i+1} = (0,b)$, $\boldsymbol{p}_i' = (0,b)$, $\boldsymbol{p}_{i+1}' = (-a,0)$ $(ab \neq 0)$ into Eq. 2 and let $\lambda_i = \mu_i = 1$, then the coordinates of CT-Hermite curve segment $\boldsymbol{H}_i(t)$ is

$$\begin{cases} x(t) = a\cos t \\ y(t) = b\sin t \end{cases}$$

This gives the intrinsic equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, it is an equation of an ellipse. Figure 5 shows an ellipse represented by CT-Hermite curve segment with $a = 1$ and $b = 2$, where solid lines is $0 \le t \le \pi/2$, dotted line is $1 \le t \le 2\pi$.

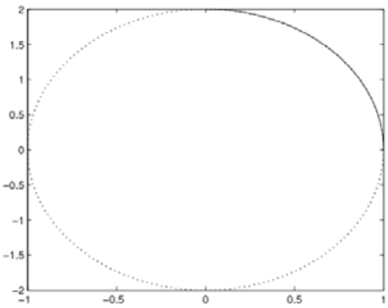

FIGURE 5 AN ELLIPSE REPRESENTED BY CT-HERMITE CURVE SEGMENT

3) Representation of a Segment of Cubic Parabola

Theorem 3. Let the endpoints and tangent vectors are $\boldsymbol{p}_i = (a,b)$, $\boldsymbol{p}_{i+1} = (0,0)$, $\boldsymbol{p}_i' = (0,0)$, $\boldsymbol{p}_{i+1}' = (-a,0)$, $ab \neq 0$, and shape parameters are $\lambda_i = \mu_i = 0$, the corresponding CT-Hermite curve segment $\boldsymbol{H}_i(t)$ represents a segment of cubic parabola .

Proof. Taking $\boldsymbol{p}_i = (a,b)$, $\boldsymbol{p}_{i+1} = (0,0)$, $\boldsymbol{p}_i' = (0,0)$, $\boldsymbol{p}_{i+1}' = (-a,0)$ $(ab \neq 0)$ into Eq. 2 and let $\lambda_i = \mu_i = 0$, then the coordinates of CT-Hermite curve segment $\boldsymbol{H}_i(t)$ is

$$\begin{cases} x(t) = a\cos t \\ y(t) = b\cos^3 t \end{cases}$$

This gives the intrinsic equation $y = \frac{b}{a^3}x^3$, it is an equation of a segment of cubic parabola. Figure 6

shows a segment of cubic parabola represented by CT-Hermite curve segment with $a=1$ and $b=2$, where solid lines is $0 \le t \le \pi/2$, dotted line is $1 \le t \le 2\pi$.

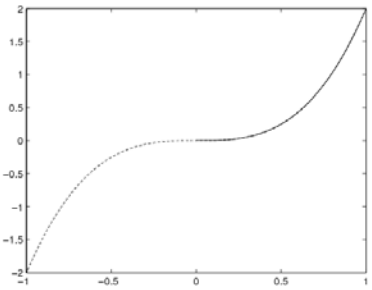

FIGURE 6 A SEGMENT OF CUBIC PARABOLA REPRESENTED BY CT-HERMITE CURVE SEGMENT

Cubic Trigonometric Hermite Parametric Surfaces

Definition 3. For $(u,v) \in [0,\pi/2]\times[0,\pi/2]$, the following surfaces

$$\boldsymbol{H}_{i,j}(u,v) = \begin{bmatrix} F_i(u) & F_{i+1}(u) & G_i(u) & G_{i+1}(u) \end{bmatrix} \times \begin{bmatrix} \boldsymbol{p}_{ij}^{00} & \boldsymbol{p}_{ij}^{01} & \boldsymbol{p}_{ij}^{v00} & \boldsymbol{p}_{ij}^{v01} \\ \boldsymbol{p}_{ij}^{10} & \boldsymbol{p}_{ij}^{11} & \boldsymbol{p}_{ij}^{v10} & \boldsymbol{p}_{ij}^{v11} \\ \boldsymbol{p}_{ij}^{u00} & \boldsymbol{p}_{ij}^{u01} & \boldsymbol{p}_{ij}^{uv00} & \boldsymbol{p}_{ij}^{uv01} \\ \boldsymbol{p}_{ij}^{u10} & \boldsymbol{p}_{ij}^{u11} & \boldsymbol{p}_{ij}^{uv10} & \boldsymbol{p}_{ij}^{uv11} \end{bmatrix} \begin{bmatrix} F_i(v) \\ F_{i+1}(v) \\ G_i(v) \\ G_{i+1}(v) \end{bmatrix}$$

are called segmented cubic trigonometric Hermite parametric surfaces (CT-Hermite surfaces in short), where

$$\mathbf{B}_{ij} = \begin{bmatrix} \boldsymbol{p}_{ij}^{00} & \boldsymbol{p}_{ij}^{01} & \boldsymbol{p}_{ij}^{v00} & \boldsymbol{p}_{ij}^{v01} \\ \boldsymbol{p}_{ij}^{10} & \boldsymbol{p}_{ij}^{11} & \boldsymbol{p}_{ij}^{v10} & \boldsymbol{p}_{ij}^{v11} \\ \boldsymbol{p}_{ij}^{u00} & \boldsymbol{p}_{ij}^{u01} & \boldsymbol{p}_{ij}^{uv00} & \boldsymbol{p}_{ij}^{uv01} \\ \boldsymbol{p}_{ij}^{u10} & \boldsymbol{p}_{ij}^{u11} & \boldsymbol{p}_{ij}^{uv10} & \boldsymbol{p}_{ij}^{uv11} \end{bmatrix}$$

are given boundary information matrix, $F_{i+j}(t)$ and $G_{i+j}(t)$ $(t=u,v;\ j=0,1)$ are CT-Hermite basis functions defined according to Eq. 1, and the shape parameters of direction u are λ_{i1} and μ_{i1}, the shape parameters of direction v are λ_{i2} and μ_{i2}.

From the properties of CT-Hermite basis functions, we can simply calculate that the CT-Hermite surfaces satisfy

$$\boldsymbol{H}_{i,j}(0,0) = \boldsymbol{p}_{ij}^{00},\quad \boldsymbol{H}_{i,j}(0,1) = \boldsymbol{p}_{ij}^{01},\quad \boldsymbol{H}_{i,j}(1,0) = \boldsymbol{p}_{ij}^{10},$$

$$\boldsymbol{H}_{i,j}(1,1) = \boldsymbol{p}_{ij}^{11},\quad \frac{\partial \boldsymbol{H}_{i,j}(0,0)}{\partial u} = \boldsymbol{p}_{ij}^{u00},\quad \frac{\partial \boldsymbol{H}_{i,j}(0,1)}{\partial u} = \boldsymbol{p}_{ij}^{u01},$$

$$\frac{\partial \boldsymbol{H}_{i,j}(1,0)}{\partial u} = \boldsymbol{p}_{ij}^{u10},\quad \frac{\partial \boldsymbol{H}_{i,j}(1,1)}{\partial u} = \boldsymbol{p}_{ij}^{u11},$$

$$\frac{\partial \boldsymbol{H}_{i,j}(0,0)}{\partial v} = \boldsymbol{p}_{ij}^{v00},\quad \frac{\partial \boldsymbol{H}_{i,j}(0,1)}{\partial v} = \boldsymbol{p}_{ij}^{v01},\quad \frac{\partial \boldsymbol{H}_{i,j}(1,0)}{\partial v} = \boldsymbol{p}_{ij}^{v10},$$

$$\frac{\partial \boldsymbol{H}_{i,j}(1,1)}{\partial v} = \boldsymbol{p}_{ij}^{v11},$$

$$\frac{\partial^2 \boldsymbol{H}_{i,j}(0,0)}{\partial u \partial v} = \boldsymbol{p}_{ij}^{uv00},\quad \frac{\partial^2 \boldsymbol{H}_{i,j}(0,1)}{\partial u \partial v} = \boldsymbol{p}_{ij}^{uv01},$$

$$\frac{\partial^2 \boldsymbol{H}_{i,j}(1,0)}{\partial u \partial v} = \boldsymbol{p}_{ij}^{uv10},\quad \frac{\partial^2 \boldsymbol{H}_{i,j}(1,1)}{\partial u \partial v} = \boldsymbol{p}_{ij}^{uv11}.$$

The above results show that CT-Hermite surfaces have the same interpolation and continuity properties with the normal cubic Hermite parametric surfaces.

Given the boundary information matrix, the shape of normal cubic Hermite curves cannot be changed, while the shape of CT-Hermite surfaces can be local or global adjusted by using the parameters in the premise of ensuring C^1 continuous.

Example 2. Given the boundary information matrix is $\mathbf{B}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$, different shape of the according CT-Hermite surface patch can be got, see Figure 7(a)-(d), where the shape parameters of (a) are $\lambda_{i1}=-2$, $\mu_{i1}=2$, $\lambda_{i2}=2$, $\mu_{i2}=-2$, (b) are $\lambda_{i1}=-1$, $\mu_{i1}=0$, $\lambda_{i2}=0$, $\mu_{i2}=1$, (c) are $\lambda_{ij}=\mu_{ij}=0$ $(j=1,2)$, and (d) are $\lambda_{i1}=2$, $\mu_{i1}=-2$, $\lambda_{i2}=-2$, $\mu_{i2}=2$.

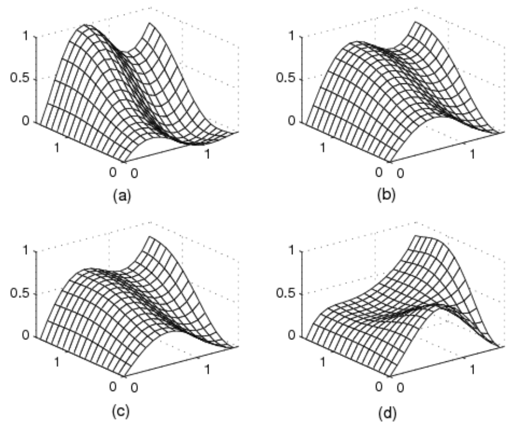

FIGURE 7 SHAPE ADJUSTMENT OF CT-HERMITE SURFACE PATCH BY SHAPE PARAMETERS

ACKNOWLEDGEMENTS

As extensions of the normal cubic Hermite parametric curves and surfaces, CT-Hermite curves and surfaces not only have the interpolation properties that normal cubic Hermite curves and surfaces have, but also have

the shape adjustable properties and the curves can exactly express some common curves in engineering. Also, because there are no differences in structure between CT-Hermite curves and the normal cubic Hermite curves, CH-Hermite surfaces and the normal cubic Hermite surfaces, it is not difficult to adapt CH-Hermite curves surfaces to a CAD/CAM system that already uses the corresponding normal cubic Hermite curves and surfaces.

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